Section 6-5
The Central Limit Theorem

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Key Concept

The procedures of this section form the foundation for estimating population parameters and hypothesis testing – topics discussed at length in the following chapters.
When selecting a simple random sample from a population
With mean \( \mu \) and standard deviation \( \sigma \),

1. If \( n > 30 \), then the sample means have a distribution that
can be approximated by a normal distribution with mean \( \mu \)
and standard deviation \( \sigma / \sqrt{n} \).

2. If \( n \leq 30 \) and the original population has a normal
distribution, then the sample means have a normal
distribution with mean \( \mu \) and standard deviation \( \sigma / \sqrt{n} \).

3. If \( n \leq 30 \) but the original population does not have a
normal distribution, then the methods of this section
do not apply.

Central Limit Theorem

Given:

1. The random variable \( x \) has a distribution (which may
or may not be normal) with mean \( \mu \) and standard
deviation \( \sigma \).

2. Simple random samples all of size \( n \) are selected
from the population. (The samples are selected so
that all possible samples of the same size \( n \) have the
same chance of being selected.)
Central Limit Theorem - cont

Conclusions:
1. The distribution of sample $\bar{x}$ will, as the sample size increases, approach a normal distribution.
2. The mean of the sample means is the population mean $\mu$.
3. The standard deviation of all sample means is $\sigma/\sqrt{n}$.

Practical Rules Commonly Used

1. For samples of size $n$ larger than 30, the distribution of the sample means can be approximated reasonably well by a normal distribution. The approximation gets better as the sample size $n$ becomes larger.
2. If the original population is itself normally distributed, then the sample means will be normally distributed for any sample size $n$ (not just the values of $n$ larger than 30).
Notation

the mean of the sample means
\[ \mu_x = \mu \]

the standard deviation of sample mean
\[ \sigma_x = \frac{\sigma}{\sqrt{n}} \]

(often called the standard error of the mean)

Simulation With Random Digits

Generate 500,000 random digits, group into 5000 samples of 100 each. Find the mean of each sample.

Even though the original 500,000 digits have a uniform distribution, the distribution of 5000 sample means is approximately a normal distribution!
Important Point

As the sample size increases, the sampling distribution of sample means approaches a normal distribution.

Example – Water Taxi Safety

Given the population of men has normally distributed weights with a mean of 172 lb and a standard deviation of 29 lb,

a) if one man is randomly selected, find the probability that his weight is greater than 175 lb.

b) if 20 different men are randomly selected, find the probability that their mean weight is greater than 175 lb (so that their total weight exceeds the safe capacity of 3500 pounds).
Example – cont

a) if one man is randomly selected, find the probability that his weight is greater than 175 lb.

\[
Z = \frac{175 - 172}{\frac{29}{\sqrt{29}}} = 0.10
\]

\[
\begin{array}{c}
0.5398 \\
0.4602
\end{array}
\]

\[
\mu_x = 172 \\
\sigma_x = 29
\]

(b) if 20 different men are randomly selected, find the probability that their mean weight is greater than 172 lb.

\[
Z = \frac{175 - 172}{\frac{29}{\sqrt{20}}} = 0.46
\]

\[
\begin{array}{c}
0.6772 \\
0.3228
\end{array}
\]

\[
\mu_x = 172 \\
\sigma_x = \frac{29}{\sqrt{20}} = 6.4845
\]
Example - cont

a) if one man is randomly selected, find the probability that his weight is greater than 175 lb.

\[ P(x > 175) = 0.4602 \]

b) if 20 different men are randomly selected, their mean weight is greater than 175 lb.

\[ P(\bar{x} > 175) = 0.3228 \]

It is much easier for an individual to deviate from the mean than it is for a group of 20 to deviate from the mean.

Interpretation of Results

Given that the safe capacity of the water taxi is 3500 pounds, there is a fairly good chance (with probability 0.3228) that it will be overloaded with 20 randomly selected men.
Correction for a Finite Population

When sampling without replacement and the sample size $n$ is greater than 5% of the finite population of size $N$, adjust the standard deviation of sample means by the following correction factor:

$$
\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} \sqrt{\frac{N - n}{N - 1}}
$$

finite population correction factor

Recap

In this section we have discussed:

- Central limit theorem.
- Practical rules.
- Effects of sample sizes.
- Correction for a finite population.